CALABI-YAU OPERATORS

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ABSTRACT. Motivated by mirror symmetry of one-parameter models, an interesting class of Fuchsian differential operators can be singled out, the so-called *Calabi–Yau operators*, introduced by ALMKVIST and ZUDILIN in [7]. They conjecturally determine Sp(4)-local systems that underly a Q-VHS with Hodge numbers

$$h^{30} = h^{21} = h^{12} = h^{03} = 1$$

and in the best cases they make their appearance as Picard–Fuchs operators of families of Calabi–Yau threefolds with $h^{12} = 1$ and encode the numbers of rational curves on a mirror manifold with $h^{11} = 1$. We review some of the striking properties of this rich class of operators.

1. CALABI-YAU OPERATORS

The story of the quintic

The story of Calabi–Yau operators is connected to the beginnings of mirror symmetry, in particular with the classical paper by CANDELAS, DE LA OSSA, GREEN and PARKES [27], which is still an excellent introduction to the subject. The larger story how mirror symmetry entered the mathematical community and has shaped a good part of present day mathematics has been told in more detail at other places, and we refer to [123, 149, 155] for nice surveys and [97, 151, 33, 91] for a more comprehensive accounts of this ever growing subject. In this paper we can only give the barest outline as far as relevant for our purpose.

Let us start with recalling the mysterious calculation with the power series

$$y_0(t) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} t^n = 1 + 120t + 113400t^2 + \ldots \in \mathbb{Z}[[t]]$$

that appeared in [27]. It represents the unique (normalised) holomorphic solution to the hypergeometric differential operator

$$\mathcal{P} := \Theta^4 - 5^5 t (\Theta + \frac{1}{5})(\Theta + \frac{2}{5})(\Theta + \frac{3}{5})(\Theta + \frac{4}{5}) \in \mathbb{Q}\left[t, \frac{d}{dt}\right],$$

where

$$\Theta := t \frac{d}{dt}$$

denotes the logarithmic derivative with respect to the parameter t. By expressing the operator in a new coordinate q, we can bring \mathcal{P} to a normal form

$$\mathcal{P} = \theta^2 \frac{5}{K(q)} \theta^2, \quad \theta := q \frac{d}{dq},$$

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where K(q) is a power series in q. In fact, it is easy to see that this q-coordinate is given by

$$q = e^{y_1(t)/y_0(t)} = t + 770t^2 + \dots,$$

where

$$y_1(t) := \log(t)y_0(t) + f_1(t), \ f_1(t) \in t\mathbb{Q}[[t]]$$

is the normalised solution of \mathcal{P} that contains a single logarithm. It is easy to compute the beginning of the power series expansion of K(q) and write it in the form of a Lambert series

$$K(q) = 5 + \sum_{d=1}^{\infty} n_d \frac{d^3 q^d}{1 - q^d}$$

from which one can read off numbers n_d , which a priori are in \mathbb{Q} . One finds

$$n_1 = 2875, \quad n_2 = 609250, \quad n_3 = 317206375, \quad \dots$$

and computing the n_d 's further, it appears that these numbers are all integers.

What makes the above calculation intriguing is the fact that it is related to the properties of two very different *Calabi–Yau threefolds*, but which the physics of strings suggests to be closely related.

A-incarnation: The first manifold is the general quintic hypersurface $X \,\subset \mathbb{P}^5$, which is a Calabi–Yau threefold with Hodge numbers $h^{11} = 1$ and $h^{12} = 101$. The numbers n_d are called *instanton numbers* and were argued in [27] to be equal to the *number of rational degree d curves on X* counted in an appropriate way. This was a big claim, as only the first two numbers were known at the time: the number 2875 of lines on a general quintic was determined by the founding father of enumerative geometry H. SCHUBERT in 1886 [141], the number of 609250 of conics was determined by S. KATZ [96] hundred years later. In a heroic *tour the force*, the number of twisted cubics on the quintic was determined by S. STRØMME and G. ELLINGSRUD [53] and in fact served as a crucial cross-check for the above calculation and resulted in the famous message: *Physics wins*! For the details of that story we refer to [156].

B-incarnation: The second manifold is the quintic mirror Y with 'flipped' Hodge numbers $h^{12} = 1, h^{11} = 101$. It was constructed via an orbifold construction that was proposed earlier by B. GREENE and R. PLESSER [71]. As the Hodge number $h^{12}(Y)$ is equal to the dimension of the local moduli space of Y, we have in fact a 1-parameter family of manifolds Y_t , parametrised by t. It can be obtained from the quintics of the so-called *Dwork pencil*

$$\sum_{i=1}^{5} x_i^5 - 5\psi \prod_{i=1}^{5} x_i = 0, \quad t = \frac{1}{(5\psi)^5}$$

by dividing out the abelian group of order 125 generated by $x_i \mapsto \zeta_i x_i, \zeta_i^5 = 1, \prod_{i=1}^5 \zeta_i = 1$ and resolving the resulting singularities. The solution to the differential equation $y_0(t^5)$ is in fact a (normalised) *period integral* of Y_t and \mathcal{P} is the associated *Picard–Fuchs equation*. The most salient property of the differential operator is the fact that it has a so-called MUM (=maximal unipotent monodromy) point at 0, where the variety Y_t degenerates to a union of divisors forming a combinatorial sphere. The computation of CANDELAS and coauthors was immediately extended to other Calabi–Yau threefolds, [122, 58, 104, 105, 115]. The extension to smooth complete intersections in weighted projective spaces yield 13 cases for which the associated differential equation is of hypergeometric type, [16]. In fact, there is a 14th case [1, 47] that was not considered at that time, as it corresponds to a Calabi–Yau variety with a singular point.

In 1993 VICTOR BATYREV came up with a general interpretation of mirror symmetry in terms of dual reflexive polytopes [13], which lead to a plethora of examples of mirror pairs of Calabi–Yau manifolds, but usually with rather large Hodge numbers. As the mirror manifold Y of X varies in a $h^{11}(X) = h^{12}(Y)$ -dimensional family, the period integrals are solutions to differential systems with this many variables. In general only in the cases that $h^{11}(X) = h^{12}(Y) = 1$, the so-called *one-parameter models*, one obtains a single ordinary differential equation of order four annihilating the period integrals. However, in [16] first examples of Calabi–Yau threefolds with $h^{11}(X) > 1$ were considered, which, due to a symmetry, still lead to a fourth order operator. A nice example is the case of the degree (3, 3) hypersurface X in $\mathbb{P}^2 \times \mathbb{P}^2$, which leads to the operator number 15 in the AESZ-list [3]

$$\Theta^4 - 3t(3\Theta + 1)(3\Theta + 2)(7\Theta^2 + 7\Theta + 2) - 72t^2(3\Theta + 5)(3\Theta + 4)(3\Theta + 2)(3\Theta + 1),$$

which no longer is of hypergeometric type. A curve on X has two degrees, coming from the two \mathbb{P}^2 -factors. The corresponding instanton numbers of the above operator count the rational curves with total degree equal to d. The mirror manifold of X has $h^{12} = 2$, but over a line in the two-dimensional deformation space the cohomology splits off a sub Hodge structure with $h^{03} = h^{12} = h^{21} = h^{30} = 1$.

The discovery that the computation of periods of one manifold provides enumerative information about another manifold was totally unexpected and left people wonder about the geometrical relation between X and Y. It was a key motivation for the development of mathematical understanding of mirror symmetry and led to several important insights. First, Gromov-Witten theory was developed to provide a rigorous basis for counting curves on general manifolds, [103]. This enabled GIVENTAL [64] and LIAN, LIU and YAU [113] to prove the mirror theorem that vindicated the above computational scheme, but left out the question of the geometrical relation between the spaces X and Y. Second, the idea that in mirror symmetry the symplectic geometry of X gets identified with the *holomorphic geometry* of Y and vice versa got a precise expression in terms of KONTSEVICH's notion of homological mirror symmetry, [100]. The insight that this in turn leads to a description of X and Y as two dual torus fibrations by STROMINGER-YAU-ZASLOW [148] took some of the mystery of the mirror symmetry phenomenon, but left the mathematical community with very difficult problems to solve. The approach of M. GROSS and B. SIEBERT seeks to develop this picture of mirror duality in the framework of algebraic geometry out of dual logarithmic degeneration data and the resulting affine manifolds with singularities, [72, 73], which can be seen as a grand generalisation of BATYREV's notion of dual reflexive polytopes.

Quantum Cohomology at the Mittag-Leffler Institute 1996–1997

In the year 1996/97 a special year on *Enumerative Geometry and its Interaction with Theoretical Physics* was organised by Geir Ellingsrud, Dan Laskov, Anders Thorup and

Stein Arild Strømme at the Mittag-Leffler Institute. The text [9] collects write-ups of the talks that were given during the first semester and capture very well the exciting atmosphere aroused by the new techniques of *Gromov-Witten theory*, *Frobenius manifolds*, *quantum cohomology*, *quantum D-modules*, all aimed at understanding the mirror theorem.

During my stay at Mittag-Leffler, I intended to steer away from toric mirror symmetry and tried to obtain further examples of one-parameter models by looking at complete intersections in other simple spaces, like homogeneous spaces. For example, complete intersections in Grassmanianns lead to varieties with $h^{11} = 1$, but as these were not of toric type, it was not so clear how to obtain a mirror dual, let alone its Picard–Fuchs equation. This had changed after DUBROVIN [49] and GIVENTAL [65] showed that it is possible to find the Picard–Fuchs equation directly from X in terms of Gromov–Witten invariants of an ambient manifold of X. More precisely, if X is a complete intersection in a manifold Z that is simple enough to allow for an explicit description of its quantum cohomology ring, one can use a *Laplace transform* to obtain the differential equation for Y: the quantum Lefschetz principle in the formulation of [65, 66].

To explain these important ideas, consider for simplicity a smooth projective variety Z with $h^2(Z) = 1$, without odd cohomology and let $H \in H^2(Z)$ be its ample generator. The homology class of a curve Σ in Z is determined by its *degree*, defined as the intersection number $\Sigma \cdot H$. For $A, B, C \in H^*(Z)$ the *Gromov-Witten three-point function* is the series

$$\langle A, B, C \rangle := \sum_{d=0}^{\infty} \langle A, B, C \rangle_d t^d,$$

where

 $\langle A, B, C \rangle_d$

is the Gromov–Witten count of rational degree d curves that meet the cycles (Poincaré dual to) A, B and C. The quantum product \star is the *t*-dependent product determined by the equation

$$(A \star B, C) = \langle A, B, C \rangle,$$

where on the left we use the non-degenerate Poincaré pairing on $H^*(Z)$. The product \star is associative and commutative and in the Fano case can be used to define a new ring structure on $QH^*(Z) := H^*(Z)[t]$: the quantum cohomology ring of Z. The *Dubrovin-Givental* connection is the connection

$$\nabla = td - H\star$$

on the trivial bundle over the t-line and with the vector space $H^*(Z)$ as fibre, whose horizontal sections S(t) are solutions to the differential system

$$\theta S = H \star S, \ \theta := t \frac{d}{dt}.$$

For $Z = \mathbb{P}^4$ the only non-trivial three-point invariant is

$$\langle H^4, H^4, H \rangle_1 = 1,$$

which expresses the obvious enumerative fact that there is a single line through two points and this line intersects a given hyperplane in a single point. The quantum cohomology ring is identified with $\mathbb{C}[H, t]/(H^5 - t)$, i.e. one has

$$H \star H = H^2, \ H^2 \star H = H^3, \ H^3 \star H = H^4, \ H^4 \star H = t$$

Using the basis $1, H, H^2, H^3, H^4$ for $H^*(\mathbb{P}^4)$, the quantum differential system for $S = \sum_i S_i H^i$ can be written as

$$t\frac{d}{dt}\begin{pmatrix}S_0\\S_1\\S_2\\S_3\\S_4\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & t\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}S_0\\S_1\\S_2\\S_3\\S_4\end{pmatrix}.$$

which leads to the differential equation of order 5 for the lowest component $S_4(t)$ of S(t):

$$(\theta^5 - t)S_4(t) = 0.$$

This equation is easily seen to have

$$\psi(t) := \sum \frac{1}{(n!)^5} t^n$$

as unique holomorphic solution. The Laplace transform of this function

$$\frac{1}{t} \int_0^\infty \psi(s^5) e^{-s/t} ds$$

is the function

$$\phi(t^5) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} t^{5n} = y_0(t^5).$$

Note that the 5's are dictated by the fact that the canonical class of \mathbb{P}^4 is K = -5H and it transforms the irregular quantum differential system into one with regular singularities!

A closely related aspect was the idea, already present in [29], that the notion of mirror symmetry should be extended from Calabi–Yau spaces to the ambient Fano manifolds and that these mirrors were described by a *Landau–Ginzburg potential*, for example in case of \mathbb{P}^4 by the Laurent polynomial

$$W = X_1 + X_2 + X_3 + X_4 + \frac{1}{X_1 X_2 X_3 X_4}.$$

The solutions to the quantum differential system of \mathbb{P}^4 have a representation as oscillatory integrals attached to W, which by Laplace transformation become period integrals of the manifold

$$\{1 - tW = 0\} \subset (\mathbb{C}^*)^4$$

that completes to a Calabi–Yau space by compactification in the toric manifold defined by the Newton polytope of W. The upshot is the following: by expanding into a geometric series, the normalised period

$$\frac{1}{(2\pi i)^4} \oint \frac{1}{1 - tW} \frac{dX_1}{X_1} \frac{dX_2}{X_2} \frac{dX_3}{X_3} \frac{dX_4}{X_4}$$

expands as

$$\sum_{n=0}^{\infty} [W^n]_0 t^n,$$

where $[-]_0$ takes the constant term of a Laurent polynomial. Indeed, for the above Laurent polynomial for the mirror of \mathbb{P}^4 one obtains

$$[W^{5n}]_0 = \frac{(5n)!}{(n!)^5},$$

which leads back to our function $y_0(t^5)$. For a much more detailed analysis of this example see [67, 77, 78].

Now for the Grassmannian Z = G(n, k) the quantum cohomology was determined by SIEBERT-TIAN, [143] and during the special year at Mittag-Leffler the idea arose to use this to calculate the Laplace transform of the quantum differential system of the Grassmannian and try to come up with predictions for the number of rational curves on the Grassmannian Calabi-Yaus. This exciting collaboration with BATYREV, KIM and CIOCAN-FONTANINE led to the papers [17] and [18].

As an example, consider the Grassmannian Z = G(2, 5); it is a six-dimensional Fano variety with K = -5H. From the quantum cohomology differential system we obtained in [17] the function

$$\psi(t) := \sum_{n=0}^{\infty} \frac{1}{n!^5} A_n t^n,$$

where

$$A_n := \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}$$

are the famous Apéry numbers related to $\zeta(2)$, [12]. The complete intersection $X := X(1,2,2) \subset Z$ of Z with three general hypersurfaces of degree 1, 2, 2 in the Plücker embedding of the Grassmannian is a Calabi–Yau threefold with $h^{11} = 1$. Using the quantum Lefschetz principle/Laplace transform we were lead to the function

$$\phi(t) = \sum_{n=0}^{\infty} \frac{n!(2n)!(2n)!}{(n!)^5} A_n t^n = \sum_{n=0}^{\infty} {\binom{2n}{n}}^2 A_n t^n$$

that should be a normalised period of a mirror Y to X(1,2,2). The function ϕ is the holomorphic solution to the operator

$$\mathcal{P} = \Theta^4 - 4t(2\Theta + 1)^2(11\Theta^2 + 11\Theta + 3) - 16t^2(2\Theta + 1)^2(2\Theta + 3)^2$$

with Riemann symbol

$$\left\{ \begin{array}{cccc} 0 & \alpha & \beta & \infty \\ \hline 0 & 0 & 0 & 1/2 \\ 0 & 1 & 1 & 1/2 \\ 0 & 1 & 1 & 3/2 \\ 0 & 2 & 2 & 3/2 \end{array} \right\},\,$$

which is number 25 in the AESZ-list [3]. The instanton numbers of X(1,2,2) were then found to be

$$n_1 = 400, n_2 = 5540, n_3 = 164400, n_4 = 7059880, \dots$$

Of particular interest is the case of the Calabi–Yau section $X := X(1, 1, 1, 1, 1, 1, 1, 1) \subset G(2, 7)$ obtained by taking 7 generic linear sections of the Grassmannian. We found

that in this case the function ψ is given as

$$\psi(t) = \sum_{n=0}^{\infty} A_n t^n,$$

where

$$A_n = \sum_{k,l} {\binom{n}{k}}^2 {\binom{n}{l}}^2 {\binom{k+l}{n}} {\binom{2n-k}{n}}.$$

At the same time at Mittag-Leffler, EINAR RØDLAND determined the mirror for the generic Pfaffian Calabi–Yau in $X' \subset \mathbb{P}^7$ via the orbifold method [134]. When he showed me the operator he had obtained, we were both electrified: it was *identical* with the above G(2,7)-operator that I had obtained a week before. But the instanton numbers for X and X' had to be different! The mystery was partly resolved after realising that the operator had *two points of maximal unipotent monodromy*. At the origin we get the instanton numbers for for the Grassmannian Calabi–Yau X,

 $n_1 = 196, n_2 = 1225, n_3 = 12740, n_4 = 198058, n_5 = 3716944, \dots$

and for the point at infinity the instanton numbers for the Pfaffian Calabi–Yau X':

 $n_1 = 588, n_2 = 12103, n_3 = 583884, n_4 = 41359136, n_5 = 360939409, \dots$

At the time we were left to wonder about the geometrical relation between $X \subset G(2,7)$ and the Pfaffian $X' \subset \mathbb{P}^7$. They are not birational, but it was shown later in [24] that the derived categories of X and X' are equivalent, as predicted by homological mirror symmetry.

Although these examples are not toric, it turns out that mirror symmetry for these examples still can be linked up with BATYREV's theory of dual reflexive polyhedra: the Grassmanianns can be *degenerated* to a toric variety with singularities in codimension 3, [17]. This leads to a Laurent polynomial description for the Grassmannian that was found before by EGUCHI, HORI and XIONG, [52]. For a beautiful recent approach to the mirror symmetry of the Grassmannian, its relation to the Langlands dual group and the cluster structure, see [118] and [132].

It was suggested at the time that one could try to invert the degeneration construction and start with special singular toric manifolds and smooth these to obtain further examples, see [14] and [8]. It has been verified that all Fano varities of dimension 2 and 3 admit such toric degenerations.

Calabi-Yau operators

In 2003 I received a letter from GERT ALMKVIST in which he asked if I knew more operators *like the one for the quintic*. Apart from the cases coming from [16] and the Grassmannian cases from [17], I knew a few more coming from the construction in [8], but soon ran out of further examples. Then, by insightful playing with various sums of binomial coefficients, ALMKVIST discovered many further examples. In the paper [7] of ALMKVIST and ZUDILIN the notion of *Calabi–Yau operator* was formulated, which is more or less characterised by the condition that the calculation of [27] works. The operators were collected in a list [3]. For a slightly more systematic and updated list, see [6] and the online database [28].

Preliminary definition: An irreducible fourth order differential operator $\mathcal{P} \in \mathbb{C}\left[t, \frac{d}{dt}\right]$ is called a *Calabi–Yau operator* if it satisfies the following conditions:

- it is of Fuchsian type.
- it is self-dual.
- it has 0 as MUM-point.
- it satisfies integrality conditions:
 - the holomorphic solution $y_0(x) \in \mathbb{Z}[[x]]$.
 - the q-coordinate $q(x) \in \mathbb{Z}[[x]]$.
 - the instanton numbers $n_d \in \mathbb{Z}$.

In fact, it is more natural to allow mild denominators and look for N-integral solution, q-coordinate and instanton numbers. Also, one could replace \mathbb{Z} by rings of integers in a number field. There is a natural notion of Calabi–Yau operator of arbitrary order that we will not spell out here. For a more thorough discussion we refer to the thesis of BOGNER [21] and [22]. Operators of order two tend to come from families of elliptic curves, those of order three are obtained, by a famous theorem of FANO [57], from those of order two by taking the second symmetric power and appear as Picard–Fuchs operators for families of K3-surfaces with Picard number equal to 19. So operators of order two and three belong to more classical realms of algebraic geometry and modular forms. The case of fourth order operators seems to be the first that leads us into completely unknown territory.

It follows from the self-duality condition of \mathcal{P} that there is a unique formal coordinate transformation $x \mapsto q = x + \ldots$ called the *mirror map* that brings the operator in the form

$$\theta^2 \frac{1}{K(q)} \theta^2,$$

where $K(q) = 1 + \ldots$ is a power series, called the normalised Yukawa coupling of \mathcal{P} . This power series is an invariant of the operator, unique up to a scaling in q. The (normalised) instanton numbers

$$n_1, n_2, n_3, \ldots \in \mathbb{Q}$$

of the operator \mathcal{P} are defined by writing the Yukawa coupling in the form

$$K(q) = 1 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1 - q^d}.$$

A general construction to obtain a self-dual fourth order operator is by taking the symmetric cube of a second order operator. But these are very special as for these operators all instanton numbers n_d vanish for d > 1 and are thus counted as *trivial*. (Although they play a role in mirror symmetry for abelian varieties, see [16] for an example.) The conditions are not all independent of each other; for example the integrality properties already *imply* the Fuchsian nature of the differential equation, [10].

There is a couple of obvious questions one can ask:

Question 1: How to construct examples of Calabi–Yau operators?

This question is partly but eloquently answered in the paper *The art of finding Calabi–Yau differential equations* by G. ALMKVIST, see [2].

We summarise here the basics. It is rather easy to fulfill the first three conditions, but the integrality conditions are much harder to satisfy.

A-incarnations: As explained above, by looking at the quantum cohomology of a Fano manifold Z one obtains a quantum differential system. If Z contains a Calabi–Yau threefold (and the Picard number is one), one obtains a Calabi–Yau operator by Laplace transformation. As we do not know the complete classification of Fano manifolds in dimension ≥ 4 one can not go far beyond a class of obvious examples, obtained from homogeneous bundles over homogeneous spaces. In fact, in [54] it was suggested that by reverse engineering one could make predictions about the existence of manifolds with given characteristic numbers from the monodromy of the operator alone.

B-incarnations: Here one is in much better shape. Take any interesting looking family of Calabi–Yau threefolds with $h^{12} = 1$ and compute the Picard–Fuchs operator. The chances are good that it will have a MUM-point somewhere. Below we describe two classes of examples we have been looking at recently. This approach is far from exhausted and there are many more constructions one can try. The recent algorithm of LAIREZ [107] for finding Picard–Fuchs operators is most useful here.

Hadamard products: If $f(t) = \sum_{n} a_n t^n$ and $g(t) = \sum b_n t^n$ are power series, the series

$$f \star g(t) := \sum a_n b_n t^n$$

obtained by taking the coefficientwise product is called the Hadamard product. A classical theorem of HADAMARD states that if f and g satisfy a Fuchsian differential equation, then so does $f \star g$. In this way quite a few Calabi–Yau operators were found. On the level of local systems, this comes down to taking the (multiplicative) convolution of the corresponding local systems. By the work of KATZ, DETTWEILER, REITER, SABBAH [98, 45, 46] the monodromy and Hodge numbers of such convolutions are under explicit control. This answers completely a question posed at the end of [54].

Binomial Sums: In many examples the coefficients a_n of the holomorphic solution are given as special binomial sums. ALMKVIST is the uncontested champion in guessing binomial sums that give Calabi–Yau operators. Using Zeilberger in MAPLE allows one to find the recursion and hence the Picard–Fuchs operator effectively.

Computer search: One can start with a parametric differential equation and make a computer search for those which give rise to cases with integral solution, mirror map and instanton numbers. This was done in [3] for operators of degree two. Going to higher degree might be possible, but is hampered by the fact that the number of free parameters becomes too big to handle by brute force.

Pullback from fifth order: A characteristic property of Calabi–Yau operators is the vanishing of a certain quantity Q (see section 2.4), which causes the second order Wronskians to satify a fifth order operator with MUM at the origin. The fourth and fifth order operator determine each other; on the level of Lie algebras this is the exceptional isomorphism $sp(4) \approx so(5)$. One starts from fifth order operators and finds by "pullback" the corresponding fourth order operator. A couple of Calabi–Yau operators were found this way, but there seem to be very few simple fifth order operators that

can be used.

The relation between fourth and fifth order operators was used with great success by ALMKVIST and GUILLERA [4, 5] to find Ramanujan type formulas for $\frac{1}{\pi^2}$, which is a formula of type

$$\sum_{n=0}^{\infty} A_n (a + bn + cn^2) z_0^n = \frac{1}{\pi^2}.$$

The first were found by GUILLERA and five of them were proved by using the WILF– ZEILBERGER machinery. It was later realized that A_n was the coefficient of hypergeometric fifth order Calabi–Yau equation. Later, using the properties of the fourth degree pullback, ALMKVIST and GUILLERA found several more formulas also for nonhypergeometric equations. A striking example is the formula

$$\sum_{n=0}^{\infty} \frac{(6n)!}{n!^6} (36 + 504n + 2128n^2) \frac{1}{1000000^n} = \frac{375}{\pi^2}$$

which in principle can be used to compute an arbitrary decimal of $\frac{1}{\pi^2}$ without computing the earlier ones.

Laurent series: This is a special case of a B-incarnation. From a Laurent polynomial W we can compute the constant term series

$$\sum_{n=0}^{\infty} [W^n]_0 t^n$$

and from it one can in turn find the Picard–Fuchs operator that annihilates it. (In fact, this was the method used in [16].) In good cases one obtains a fourth order operator with MUM. In [15] BATYREV and KREUZER produced a list of promising candidate Laurent polynomials. Some new Calabi–Yau operators were found in this way.

The group of CORTI, COATES, and KASPRZYK from Imperial College in London has been persuing this approach on a larger scale, systematically using all reflexive polytopes and Laurent polynomials with special choice of the coefficients, [31]. From a preliminary run [32], 19 new operators were found and one can reasonably expect many more to come from this approach.

Diagonals: Not all Calabi–Yau operators arise from Laurent polynomials. One obstruction comes from the fact that the numbers

$$a_n = [W^n]_0$$

coming from a Laurent polynomial satisfy *Dwork congruences*, [137] and [121]. The simplest of these imply that a_n satisfy for each prime number p the congruence

$$a_{n_0+n_1p+\dots n_kp^k} = a_{n_0}a_{n_1}\dots a_{n_k} \mod p.$$

A more general concept is that of a *diagonal*. If

$$f = \sum a_{k_1 k_2 \dots k_n} X_1^{k_1} X_2^{k_2} \dots X_n^{k_n} \in \mathbb{Q}[[X_1, X_2, \dots, X_n]]$$

is a power series in n variables, then the diagonal $\Delta_n(f)$ of f is the power series in one variable obtained by only retaining the diagonal coefficients:

$$\Delta_n(f) := \sum_{k=0}^{\infty} a_{kk\dots k} t^k \in \mathbb{Q}[[t]].$$

As was shown by CHRISTOL [30], the diagonals of rational functions $P/Q, Q(0) \neq 0$ always satisfy a Fuchsian differential equation which is of geometric origin. More generally, by LIPSCHITZ [116] the diagonal of any *D*-finite (i.e. holonomic) series is *D*-finite.

Note that if W is a Laurent polynomial in X_1, X_2, \ldots, X_n , then its constant term series is a diagonal of a rational function:

$$\sum_{n} [W^{n}]_{0} t^{n} = \Delta_{n+1} \left(\frac{1}{1 - X_{0} X_{1} \dots X_{n} W(X_{1}, X_{2}, \dots, X_{n})} \right)$$

Any binomial sum can be converted into a representation as the diagonal of a rational function [25], hence the corresponding differential operator is always of geometrical origin in the sense of [10].

In these constructions *integrality of the solution* is put in by construction, but the integrality of the mirror map and instanton numbers is for most operators *conjectural* and an experimental fact only.

Question 2: How many Calabi–Yau operators do exist? Is their number finite or infinite?

Of course, two operators that are related by a coordinate transformation or by multiplication with an algebraic function are to be considered as equivalent and we should count classes. Clearly, this is related to the question if there are finitely many or infinitely many distinct topological types of Calabi–Yau threefolds, one of the big mysteries of the subject. It is not clear what to expect nor what to hope for.

A Calabi–Yau operator can be written in Θ -form as

$$\mathcal{P} := \Theta^4 + tP_1(\Theta) + t^2 P_2(x) + \ldots + t^r P_r(\Theta),$$

where the P_k are polynomials of degree four in Θ and we assume $P_r \neq 0$. The number r is then called the *degree* of the Calabi–Yau operator \mathcal{P} . Over the last 12 years ALMKVIST, myself and others have been busy with collecting, simplifying and sorting operators by degree, which is the simplest measure of complexity. Our most recent list (August 2016) contains the the following operators:

degree	1	2	3	4	5	6	7	8	9	10
number of cases	14	70	36	77	134	42	19	84	12	10
degree	11	12	13	14	15	16	17	18	19	20
number of cases	20	17	6	9	0	16	0	0	0	1
deg	gree	21	22	23	24		32		40	
number of ca	2	0	0	17		1		2	1	

Note that we are counting operators together with the choice of a MUM-point. Some operators have more than one MUM-point, so these make, in transformed form, multiple appearance on the list. All listed cases are really different, as the instanton numbers are different. However, it is conceivable that some of the operators of high degree are transformable to ones of lower degree, which would change the above table correspondingly. The operators are collected in a database that is accessible online, [28].

Question 3: For which operators do exist Calabi–Yau incarnations?

One might ask: does there exist an A-incarnation for a given operator \mathcal{P} ? That is, does there exist a Calabi–Yau threefold X with $h^{11} = 1$ for which the instanton numbers are the instanton numbers of the operator?

$$n_d(X) = n_d(\mathcal{P})?$$

For this we have only few examples and there are many operators which the existence of such a manifold is in serious doubt. For so-called *conifold operators* (see section 2.6) one can compute characteristic numbers like the Euler number from the monodromy of the operator, [54]. However, there are a number of cases where this number turns out to be *positive*. See [44] for an example worked out in detail.

Does there exist a *B*-incarnation for \mathcal{P} ? That is, does there exist a Calabi–Yau threefold *Y* with $h^{12} = 1$ for which the Picard–Fuchs operator is \mathcal{P} ? If this happens, we say that \mathcal{P} has a strong *B*-incarnation (if *Y* is even projective we call it a very strong *B*-incarnation). One could ask the operator to be a right factor of the Picard–Fuchs of a Calabi–Yau variety with $h^{12} > 1$, which might be called a weak *B*-realisation. Differential operators having an integral solution are *G*-operators in the sense of [10]; conjectures of BOMBIERI and DWORK say that such an operator is of geometric origin. For the cases where the coefficients a_n have a representation as a binomial sum it is a theorem that they are of geometric origin.

Even if for some operators there do not exist strict A- or B-incarnations, it seems fruitful to consider each member of the list as describing something like a rank four Calabi-Yau motive over \mathbb{P}^1 and try to reconstruct as much as possible of the geometry out of the differential operator alone.

Some recent examples

We report on two classes of examples of Calabi–Yau threefolds with $h^{12} = 1$ that are geometrically accessible and exhibit various interesting phenomena. These examples will be discussed in two forthcoming papers with CYNK [37] and [38]. The computation of the corresponding Picard–Fuchs operators became possible using the program of LAIREZ, [107].

Double octics. A *double octic* is a threefold Y that arises as the double cover of \mathbb{P}^3 ramified over a surface $D \subset \mathbb{P}^3$ of degree eight.

If D is smooth, Y is a smooth Calabi–Yau threefold with Hodge numbers $h^{11} = 1$, $h^{12} = 149$. If D has singularities, Y is singular as well, but sometimes admits a crepant resolution. Of particular interest is the case where D is the union of eight planes: as long as there are no fivefold points or fourfold lines in the configuration of planes, there exists a (projective) crepant resolution \hat{Y} that can be obtained as covering of the blow-up of \mathbb{P}^3 . By [34], the infinitesimal deformations of \hat{Y} can be identified with the equisingular deformations of the divisor D, which thus can be read of from the combinatorics of the intersection pattern of the planes. In the thesis of MEYER [120], 63 families of such double octics with $h^{12}(\hat{Y}) = 1$ were identified. We determined for all these cases the corresponding Picard–Fuchs operator and some new operators were found this way. A particular beauty is the Calabi–Yau obtained from arrangement number 254 of MEYER. The octic D is defined by the equation

$$xyzu(x + y + z + u)(u + y + tz)(zt + tu + x + y)(x + ty + zt) = 0,$$

where x, y, z, u are coordinates on \mathbb{P}^3 and t is the parameter. The Picard–Fuchs operator is a rather complicated operator of degree 12 with Riemann symbol

ſ	0	1/2	1	α	β	a	b	c	∞))
	0	0	0	0	0	0	0	0	3/2	
	0	1	0	1	1	1	1	1	3/2	Y
	0	1	0	1	1	3	3	3	3/2	
	0	2	0	2	2	4	4	4	3/2	

The operator has *conifold points* at $1/2, \alpha, \beta$, where α, β are roots of $t^2 - 3t + 1 = 0$, and *apparent singularities* (so the local monodromy is trivial here) at a, b, c, roots of $8t^3 - 10t^2 + t - 1 = 0$. At 0, 1 and at ∞ (after a quadratic pullback) we have MUM points, with *three* different sets of instanton numbers.

	0	1	∞
n_1	288	128	4
n_2	59200	-4796	7/2
n_3	-8252768	341632	52
n_4	-1223488576	-31623118	500
n_5	585571467872	3395329408	2796

On the A-side we expect *three* birationally distinct Calabi–Yau geometries with these (normalised) instanton numbers, which have equivalent derived categories!

In many other cases we obtain strong B-incarnations of operators that were known before. Also, there are many so-called *orphans* (see also section 2.5) and there are cases where the Picard–Fuchs operator is of order two. For details we refer to the forthcoming [37].

It appears that there exist many operators in the list that have *two* points of maximal unipotent monodromy. In a recent series of papers [88, 89, 90], HOSONO and TAKAGI described the beautiful geometry of the *Reye congruence Calabi–Yau threefold X*. The symmetric complete intersection of five divisors of degree (1, 1) in $\mathbb{P}^4 \times \mathbb{P}^4$ was considered in [16], but the Reye Calabi–Yau threefold X arises from this complete intersection by dividing out the involution interchanging the \mathbb{P}^4 -factors and has $h^{11} = 1, h^{21} = 26$. The corresponding Picard–Fuchs operator for the mirror family was described in [16] and appeared as number 22 in the AESZ-list [3]:

$$2^{4}x^{4} \left(465\theta^{4} + 1266\theta^{3} + 1439\theta^{2} + 806\theta + 184\right) + 2^{9}x^{5} \left((\theta + 1)^{4}\right)$$

It has a second MUM-point at infinity, whose mirror appears to be the *double quintic* symmetroid X', a double cover of the general linear symmetric determinant in \mathbb{P}^4 . So this is a second example quite similar to that of the Pfaffian and the Grassmannian Calabi–Yau and indeed in [90] X and X' were shown to be derived equivalent.

Elliptic fibre products. By blowing up the nine intersection points of a pencil of plane cubics we obtain a rational elliptic surface \mathcal{E} . By construction it admits a map $\pi : \mathcal{E} \longrightarrow \mathbb{P}^1$ and the fibres are identified with the cubics of the pencil; from the Euler number $\chi(\mathcal{E}) = 3 + 9$ we see that in general there will be 12 nodal cubics in the family. As all cubics through eight of the base points also pass through the nineth, and four of the eight points can be fixed, we see that the construction depends on eight parameters, and thus that there is one condition on the position of the 12 singular fibres of a rational elliptic surface. By specialisation of the construction, the singularities of these fibres may coalesce to form other Kodaira types, but the sum of their Euler numbers will always add up to 12. There are lists by SCHMICKLER-HIRZEBRUCH [140] and by HERFURTNER [80] that give all possible combinations of three and four Kodaira fibres; among them there are the six *Beauville surfaces* [19] with four fibres of type I_n . In 1988 SCHOEN [138] decribed a simple and very interesting class of Calabi–Yau threefolds by taking the fibre product of two such rational elliptic surfaces \mathcal{E}_i , i = 1, 2:

$$Y := \mathcal{E}_1 \times_{\mathbb{P}^1} \mathcal{E}_2 \longrightarrow \mathbb{P}^1.$$

If the sets of singular values $\Sigma_i \subset \mathbb{P}^1$ of \mathcal{E}_i are disjoint, then Y is a smooth Calabi–Yau threefold that depends on 19 = 11 + 11 - 3 parameters. The Euler number is equal to zero, as the fibres over a point of \mathbb{P}^1 all have Euler number zero, and indeed the Hodge numbers of Y are $(h^{11}, h^{12}) = (19, 19)$. If, however, the fibrations $\mathcal{E}_i \longrightarrow \mathbb{P}^1$ have singular points in common, the threefold Y aquires singularities. For example, when an I_n fibre meets an I_m fibre, Y aquires $n \cdot m$ singularities of type A_1 . When we take a small resolution \hat{Y} of these singularities, we obtain a smooth Calabi–Yau threefold whose Hodge numbers can be determined easily from the singular fibres. In particular, there is a large number of cases where $h^{12}(\hat{Y}) = 1$, involving elliptic surfaces with up to six singular fibres. In [35] we started exploring these examples, but it was only after LAIREZ's program [107] became available that we were able to determine the most complicated of the corresponding Picard–Fuchs operators.

Example: We take for \mathcal{E}_1 the Beauville surface with fibres I_6 , I_3 , I_2 , I_1 and as \mathcal{E}_2 a surface with five singular fibres I_8 , I_1 , I_1 , I_1 , I_1 that depends on a single modulus-parameter t (the Weierstrass equation for this surface is too complicated to write down here). We identify the bases of the fibrations of \mathcal{E}_1 and \mathcal{E}_2 in such a way that three of the fibres of these two families appear over the same point of \mathbb{P}^1 in the following way:

	0	1	∞			
\mathcal{E}_1	I_6	I_3	I_2	I_1	_	_
\mathcal{E}_2	I_8	I_1	I_1	—	I_1	I_1

For generic choice of the modulus parameter t, the three "free" I_1 fibres will be disjoint and we obtain a Calabi–Yau threefold with $8 \cdot 6 + 3 \cdot 1 + 2 \cdot 1 = 53 A_1$ -singularities. A small resolution of these is a (non-projective) Calabi–Yau threefold \hat{Y} with Euler characteristic $2 \cdot 53 = 106$, so the Hodge numbers are $(h^{11}(\hat{Y}), h^{12}(\hat{Y})) = (52, 1)$. If the modulus-parameter t is varied, the free I_1 -fibres move, and for certain values further collisions of fibres do occur, leading to varieties with other types of singularities. The Picard–Fuchs operator obtained is the most complex one encountered up to now. It made up number 500 in the list and was presented to ALMKVIST on occasion of his 3⁴ birthday, [36]. Currently, we are exploring the properties of the operators that can be obtained from this rich class of examples, [38].

2. Some background

In the second part of this paper we explain in some more detail certain of the concepts that were freely used in the first part.

2.1. Differential equations of Fuchsian type. We first briefly go over the basic properties of linear differential operators on the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ relevant for our discussion. We refer to GRAY [69] for an overview of the historical development of this very rich subject. The classical book of INCE [92] contains a treasure of information and is still worth reading.

The set of singularities $\Sigma \subset \mathbb{P}^1$ of a differential operator

$$\mathcal{P} := a_n(t)\frac{d^n}{dt^n} + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}}\dots + a_0(t) \in \mathbb{C}[t, \frac{d}{dt}]$$

are the zeros of $a_n(t)$ together, possibly, with the point ∞ . In a neighbourhood of each point $p \notin \Sigma$ one find a basis of holomorphic solutions to the differential equation. A singular point $p \in \Sigma$ is said to be *regular singular* if all solutions grow, at radial approach, at most as a power of the inverse distance to p. FUCHS [70] found a simple condition for this to happen: 0 is a regular singular point of \mathcal{P} if and only if $ord_0(a_i/a_n) \leq n-i$. Differential equations with only regular singular points are called *regular singular* or of *Fuchsian type*. The solutions to such equations have an expansion of the form

$$\sum a_{\alpha,k} t^{\alpha} \log^k(t)$$

that is convergent on a slit disc around each singular point. The *Riemann symbol* of an operator summarises the information about the local behaviour of the solutions near the singular points of a differential operator. It consists of a table with columns indexed by the singular points under which the corresponding *exponents* are written. These exponents determine the local behaviour of the solutions at a singular point and can be determined as follows. If the operator \mathcal{P} is written in $\Theta = t \frac{d}{dt}$ -form

$$\mathcal{P} := P_0(\Theta) + tP_1(\Theta) + t^2 P_2(\Theta) + \ldots + t^d P_d(\Theta),$$

then the exponents of \mathcal{P} at 0 are just roots of the polynomial $P_0(\Theta)$. The exponents of \mathcal{P} at an arbitrary point p are obtained by first translating the point p to the origin 0 and writing the transformed operator in Θ -form and reading off the new P_0 . The exponents of \mathcal{P} at $p = \infty$ are obtained using the reciprocal transformation $t \mapsto 1/t$. When \mathcal{P} is given in Θ -form, this is very easy operation, as it amounts to reversing the sequence of polynomials P_0, P_1, \ldots, P_d and replacing Θ by $-\Theta$, so that the exponents of \mathcal{P} at infinity are given by the negatives of the roots of the polynomial P_d . 2.2. **Picard–Fuchs equations.** The differential equations that arise in algebraic geometry are usually those that are satisfied by integrals of rational or algebraic functions that depend on parameters and are called *Picard–Fuchs equations*. It was realised early on that such differential equations do have regular singularities. According to HOUZEL [84], probably the first example was the equation found by EULER [55], that describes the circumference of an ellipse with semi-axes of length 1 and $\sqrt{1-t^2}$ as function of its eccentricity t:

$$I(t) := 4 \int_0^1 \sqrt{\frac{1 - t^2 x^2}{1 - x^2}} dx = 2\pi \left(1 - \left(\frac{1}{2}\right)^2 t^2 - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{t^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{t^6}{5} - \dots \right),$$

which became later know as a *complete elliptic integral of the second kind*. From the series expansion the differential equation satisfied by I(t) is easily found to be

$$\mathcal{P} = \Theta^2 - t^2(\Theta - 1)(\Theta + 1)$$

which has

$$\left\{ \begin{array}{cccc} 0 & 1 & -1 & \infty \\ \hline 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{array} \right\}$$

as Riemann symbol. Another early example is the period of the mathematical pendulum (L = g = 1) with initial angle ϕ as function of $t := \sin(\phi/2)$:

$$4\int_0^1 \frac{1}{\sqrt{(1-x^2)(1-t^2x^2)}} dx = 2\pi \left(1 + \left(\frac{1}{2}\right)^2 t^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 t^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 t^6 + \dots\right),$$

which is an elliptic integral of the first kind and satisfies the *Legendre differential* equation

$$\Theta^2 - t^2(\Theta + 1)^2$$

with Riemann symbol

$$\left\{ \begin{array}{rrrr} 0 & 1 & -1 & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right\}.$$

After initial work by POINCARÉ and PICARD [127], it was with the work of DWORK [50] and GRIFFITHS [74], [75] that integrals on higher dimensional manifolds were studied systematically and methods to determine the Picard–Fuchs equation were developed. The general geometric setting is most conveniently formulated as follows. One starts with a projective family

$$f: \mathcal{Y} \longrightarrow \mathbb{P}^1$$

and let $\Sigma \subset \mathbb{P}^1$ be the set of critical values of f, so that f is a smooth map when restricted to $\mathbb{P}^1 \setminus \Sigma$. For $t \in \mathbb{P}^1 \setminus \Sigma$ the fibre $Y_t := f^{-1}(t)$ is a smooth d-dimensional variety; for $t \in \Sigma$ the fibre Y_t will aquire a singularity. Now choose for $p \in \mathbb{P}^1 \setminus$ Σ a d-cycle $\gamma_p \in H_d(Y_p)$. Using the local topological triviality of f over $\mathbb{P}^1 \setminus \Sigma$, we can transport γ_p to neighbouring fibres and obtain cycles $\gamma_t \in H_d(Y_t)$ for t in a neighbourhood of p. If we choose a relative differential d-form

$$\omega \in \Gamma(\mathcal{Y}, \Omega^d_{\mathcal{Y}/\mathbb{P}^1})$$

one can form the *period integral*

$$\phi(t) = \int_{\gamma_t} \omega_{|Y_t},$$

which initially is defined in a neighbourhood of p, but which can, using the local topological triviality of f, be extended along arbitrary paths in $\mathbb{P}^1 \setminus \Sigma$. The finite dimensionality of the cohomology space $H^d_{dR}(Y_t)$ implies that ϕ satisfies a linear differential equation, called the *Picard–Fuchs equation*. If the family $f : \mathcal{Y} \longrightarrow \mathbb{P}^1$ is given in terms of polynomial equations it is in principle always possible to find this Picard–Fuchs equation, but it might not be simple to do so in practice. For recent computer implementations see [124] and [107].

2.3. Local systems and monodromy. If \mathcal{P} is a differential operator of order n with set of singularities Σ , then the set of solutions to $\mathcal{P}\phi = 0$ form, in the neighborhood of each regular point p, a \mathbb{C} -vector space \mathbb{L}_p of dimension equal to n: at p a solution is uniquely determined by the values of the first n-1 derivatives. Hence, we obtain a local system of solutions $\mathbb{L} := Sol(\mathcal{P})$ on $\mathbb{P}^1 \setminus \Sigma$. If we choose a base point $p \in \mathbb{P}^1 \setminus \Sigma$, the local system determines and is determined by a representation of the fundamental group

$$T: \pi_1(\mathbb{P}^1 \setminus \Sigma, p) \longrightarrow Aut(\mathbb{L}_p) \simeq Gl_n(\mathbb{C}), \ \gamma \mapsto T_\gamma$$

called the *monodromy representation* of \mathcal{P} , which describes the behavour of solutions of \mathcal{P} under analytic continuation along closed paths. The image of the fundamental group under T is called the *monodromy group* of \mathcal{P} . A choice of generators $\gamma_1, \gamma_2, \ldots, \gamma_r$ of $\pi_1(\mathbb{P}^1 \setminus \Sigma, p)$ results in a r-tuple of matrices $T_i := T_{\gamma_i}$

$$(T_1, T_2, T_3, \ldots, T_r) \in GL_n(\mathbb{C})^r$$

that completely describes the local system. A change of base in \mathbb{L}_p leads to a simultaneous conjugation of all T_i . These powerful ideas were introduced by RIEMANN [131] in his study of the classical hypergeometric function $F(\alpha, \beta, \gamma; t)$. In fact he determined the monodromy representation for the classical hypergeometric differential operator and showed that it characterised the equation. In his thesis, LEVELT [111] found the monodromy representation of the higher hypergeometric functions. In fact, in this case the representation is uniquely determined by the Jordan type of the local monodromies around the singular points and thus represent the simplest examples of what is now called a *rigid local system*, [98].

If the operator \mathcal{P} is Fuchsian, the Zariski closure of the monodromy group is equal to the *differential Galois group Gal*(\mathcal{P}) that is introduced in the theory of PICARD and VESSIOT in analogy with the Galois group of an algebraic equation. We refer to the book [130] for a detailed account.

In his famous 1900 ICM address held in Paris, HILBERT asked in his 21. problem for the existence of a Fuchsian differential operator with prescribed monodromy representation. This is the so-called *Riemann-Hilbert problem* and is of central importance in contemporary mathematics; its solution is an interesting chapter in the history of mathematics, [128, 129, 20, 11, 39].

In the case of Picard–Fuchs operators, one starts with a projective family

$$f: \mathcal{Y} \longrightarrow \mathbb{P}^1,$$

smooth over $\mathbb{P}^1 \setminus \Sigma$ and considers the direct image sheaf

$$R^d f_* \mathbb{C}_{\mathcal{Y}}$$
.

It restricts to a local system \mathbb{H} on $\mathbb{P}^1 \setminus \Sigma$ and one has

$$\mathbb{H}_t = H^d(Y_t, \mathbb{C}).$$

In fact, the local system \mathbb{H} has many further special properties. For instance, we can consider the restriction $\mathbb{H}_{\mathbb{Z}}$ of $R^d f_* \mathbb{Z}_{\mathcal{Y}}$ to $\mathbb{P}^1 \setminus \Sigma$ and we have

$$\mathbb{H}_{\mathbb{Z}}\otimes\mathbb{C}=\mathbb{H}$$

so $\mathbb{H}_{\mathbb{Z}}$ produces a lattice bundle inside \mathbb{H} and so the monodromy representation lands in $GL_n(\mathbb{Z})$. The behaviour of cycles under parallel transport was studied by PICARD [127] and LEFSCHETZ [110] and these works represent the first topological studies of higher dimensional algebraic varieties. It led to the *Picard–Lefschetz* formula

$$T_{\gamma}: H^d(Y_p) \longrightarrow H^d(Y_p), v \mapsto v \pm \langle v, \delta \rangle \delta$$

describing the cohomological monodromy that a cycle v undergoes under parallel transport along a path γ that circumscribes (in the positive direction) a singular fibre that aquires a node (A₁-singularity) (see also [108], [117]); δ is the vanishing cycle: the class of a sphere that gets contracted when passing to the singular fibre. In general, all sorts of singularities might appear in the fibres. It is a fundamental fact that the local monodromy transformation around any singular point s

$$T_s: H^d(Y_t) \longrightarrow H^d(Y_t)$$

is always quasi-unipotent: there exist m, k such that

$$(T_s^m - I)^k = 1.$$

(In fact, one can take k = d + 1.) This is called the *monodromy theorem* and was first proven in [109]. As a consequence, the exponents of Picard–Fuchs operators are always rational.

2.4. **Self-duality.** The local systems \mathbb{H} coming from geometry also have a build-in self-duality that reflects *Poincaré duality* in the fibres: intersection of cycles in the fibres Y_t defines a non-degenerate pairing

$$\mathbb{H}_t \times \mathbb{H}_t \longrightarrow \mathbb{C}.$$

This leads to a self-duality property of the local system \mathbb{H} and of the corresponding Picard–Fuchs operators \mathcal{P} .

Recall that the *adjoint* \mathcal{P}^* of a differential operator is obtained by reading the operator backwards with alternating signs: if

$$\mathcal{P} = \sum_{i=0}^{n} a_i(t) \frac{d^i}{dt^i} \in \mathbb{Q}(t) \left[\frac{d}{dt} \right]$$

then

$$\mathcal{P}^* = \sum_{i=0}^n \left(-\frac{d}{dt}\right)^i a_i(t) \in \mathbb{Q}(t) \left[\frac{d}{dt}\right].$$

This notion was introduced by FROBENIUS in [60].

We say that an operator \mathcal{P} is *essentially self-adjoint* if there exists a function $\alpha \neq 0$ such that

$$\mathcal{P}\alpha = \alpha \mathcal{P}^*$$

If such α exists, it is easy to see that it has to satisfy the differential equation

$$\alpha' = -\frac{2}{n}a_1\alpha.$$

So if the residues of a_1 are rational, α will be an algebraic function and the local system of solutions to an essentially self-adjoint operator has a non-degenerate pairing with values in a rank one local system defined by α . It is symmetric if n is odd and alternating if n is even.

Only in the case that α is a rational function, we get an honest pairing and the differential Galois group $Gal(\mathcal{P})$ is contained in Sp(n) (*n* even) or SO(n) (*n* odd).

For a fourth order differential operator

$$\mathcal{P} := \frac{d^4}{dt^4} + a_3(t)\frac{d^3}{dt^3} + a_2(t)\frac{d^2}{dt^2} + a_1(t)\frac{d}{dt} + a_0(t) \in \mathbb{Q}(t)\left[\frac{d}{dt}\right]$$

the quantity

$$Q := \frac{1}{2}a_2a_3 - a_1 - \frac{1}{8}a_3^3 + a_2' - \frac{3}{4}a_3(a_3)' - \frac{1}{2}a_3''$$

was introduced in [7] and taken as part of the definition of the notion Calabi–Yau operator. It was shown in [7] that the vanishing of Q is equivalent to the fact that the second order Wronskians of \mathcal{P} satisfy an equation of order five rather than six and this is equivalent to \mathcal{P} being essentially self-adjoint in the above sense. As a consequence, the condition Q = 0 does not always lead to operators with $Gal(\mathcal{P}) \subset Sp(4)$, but only so after going to a cover defined by the algebraic function α (which, in fact, is the unnormalised Yukawa coupling expressed in the original coordinate). An example is the operator number 245 from the AESZ-list [3]

$$\mathcal{P} = \Theta^4 - t(216\Theta^4 + 396\Theta^3 + 366\Theta^2 + 168\Theta + 30) + 36t^2(3\Theta + 2)^2(6\Theta + 7)^2$$

with instanton numbers

$$n_1 = -6$$
, $n_2 = -33$, $n_3 = -170$, $n_4 = -1029$, $n_5 = -3246$

for which α is

$$\alpha(t) = \frac{1}{t^3(1 - 108t)^{11/6}}.$$

So only after going to a sixfold cover we do obtain an Sp(4)-operator. We note that α is a rational function if and only if the exponents at all singular points add up to an even integer, see e.g.[22]

2.5. **MUM and Hodge theory.** We say the operator \mathcal{P} has a MUM point at 0, if written in Θ -form we have

$$\mathcal{P} = \Theta^4 + tP_1(\Theta) + \dots$$

In this case, the vector space H_0 of solutions on an (arbitrary small) slit disc around the origin has a very special basis of solutions, called the *Frobenius basis*

$$\begin{array}{rcl} y_0(t) &=& f_0(t) \\ y_1(t) &=& \log(t)y_0(t) + f_1(t) \\ y_2(t) &=& \frac{1}{2}\log(t)^2y_0(t) + \log(t)y_1(t) + f_2(t) \\ y_3(t) &=& \frac{1}{6}\log(t)^3y_0(t) + \frac{1}{2}\log(t)^2y_1(t) + \log(t)y_0(t) + f_3(t), \end{array}$$

where the f_i are convergent power series with $f_0(0) = 1$, $f_i(0) = 0$, i = 1, 2, 3. We will also use the scaled Frobenius basis

$$u_{3-k} := y_k/(2\pi i)^k, \quad k = 0, 1, 2, 3$$

The local monodromy around 0 on the four dimensional vector space

$$H_0 = \langle u_0, u_1, u_2, u_3 \rangle$$

with respect to this scaled Frobenius basis is given by the matrix

$$T_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1/2 & 1 & 1 & 0 \\ 1/6 & 1/2 & 1 & 1 \end{pmatrix},$$

which is unipotent with a Jordan block of maximal possible size, hence maximal unipotent monodromy, that is for short, MUM.

The local systems \mathbb{H} on $\mathbb{P}^1 \setminus \Sigma$ that arise from algebraic geometry have strong additional properties: the cohomology of the fibre $H^d(Y_t)$ carries a pure Hodge structure of weight d: for each t we have a Hodge decomposition

$$H^{d}(Y_{t}) = \sum_{p+q=d} H_{t}^{p,q}, \ H_{t}^{p,q} = H^{q}(Y_{t}, \Omega^{p})$$

that in fact depends nicely on t: the spaces of the Hodge filtration

$$F_t^k = \sum_{p \ge k} H_t^{p,q}$$

form holomorphic vector bundles on $\mathbb{P}^1 \setminus \Sigma$

$$\mathcal{F}^d \subset \mathcal{F}^{d-1} \subset \ldots \subset \mathcal{F}^0 = \mathbb{H} \otimes \mathcal{O}_{\mathbb{P}^1 \setminus \Sigma}.$$

One says that the local system \mathbb{H} on $\mathbb{P}^1 \setminus \Sigma$ underlies a variation of Hodge structures (VHS). It is a fundamental fact proven by SCHMIDT [139] that one may extend this structure defined on $\mathbb{P}^1 \setminus \Sigma$ over the punctures $s \in \Sigma$ to a mixed Hodge structure (MHS): for each $s \in \Sigma$ there is a Q-vector space $H_s = H^d_{\lim}(Y_s)$ of dimension equal to the rank of the local system \mathbb{H} . It can be defined as the sections of the Q-local system over an arbitrary small slit disc centered at s. Write the local monodromy as $T_s = U_s S_s$, where U_s is unipotent and S_s is semi-simple, and define the monodromy logarithm as

$$N_s = -\log U_s = (1 - U_s) + \frac{1}{2}(1 - U_s)^2 + \frac{1}{3}(1 - U_s)^3 + \dots$$

The nilpotent endomorphism N_s defines a weight filtration

$$W_0 \subset W_1 \subset \ldots \subset W^{2d} = H_s,$$

which is characterised by the property that

$$N_s^k: Gr_{d+k}^W \xrightarrow{\simeq} Gr_{d-k}^W.$$

One can use the Hodge filtration \mathcal{F}^{\bullet} to define a limit Hodge filtration F_s^{\bullet} on H_s and the fundamental theorem is that for each $s \in \Sigma$ the triple $(H_s, W_{s,\bullet}, F_s^{\bullet})$ is a mixed Hodge structure: the filtration F_s^{\bullet} defines a pure Hodge structure of weight k on the graded pieces $Gr_k^W H_s$. In the geometrical case STEENBRINK [146] has constructed this mixed Hodge structure on H_s using a semi-stable model

$$\begin{array}{cccc} D & \hookrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \{s\} & \hookrightarrow & \Delta \end{array}$$

over a disc Δ . Here D is a (reduced) normal crossing divisor with components D_i . The complex of relative logarithmic differential forms

$$\Omega^{\bullet}_{\mathcal{Z}/\Delta}(\log D)$$

can be used to describe the cohomology of the fibres and its extension to Δ . The complex comes with two filtrations F^{\bullet} , W_{\bullet} , which induces filtrations on the hypercohomology groups

$$\mathbb{H}^d(\Omega^{\bullet}_{\mathcal{Z}/\Delta}(\log D)\otimes \mathcal{O}_D),$$

which then leads to the limiting mixed Hodge structure on H_s . We refer to [126] for a detailed account. Of particular relevance is the resulting *weight spectral sequence*, which expresses the graded pieces $Gr_k^W H_s$ in terms of the intersection pattern of the exceptional divisors and which degenerates at E_2 . The E_1 -term is given by

$$E_1^{p,q} := \sum_k H^{q+2(p-k)}(D[2p-k])$$

= $H^{q+2p}(D[2p]) \oplus H^{q+2p-2}(D[2p-1]) \oplus \dots,$

where

$$D[k] := \coprod D_{i_0} \cap D_{i_1} \cap \dots D_{i_k}$$

and where the sum runs over all indices $i_0 < i_1 < \ldots < i_k$. In the diagram below, the stars indicate possible non-zero entries in the E_1 -page of the weight spectral sequence of a degeneration of a threefold.

*	*	*			
*	*	*			
*	*	*	*		
	*	*	*		
	*	*	*	*	
		*	*	*	
		*	*	*	*

The differential runs horizontally, the operator N acts on it and goes two steps to the right and two steps down. There is a reflection symmetry around the central point.

The bottom row $E_1^{p,0}$ can be identified with the complex

$$0 \longrightarrow H^0(D[0]) \longrightarrow H^0(D[1]) \longrightarrow H^0(D[2]) \longrightarrow \ldots \longrightarrow H^0(D[d]) \longrightarrow 0$$

where the differential is induced by the inclusion maps. The *dual intersection complex* Γ has 0-cells in bijection to the irreducible components of D, 1-cells in bijection to the intersections of divisors, etc. So we see that the bottom row complex computes the cohomology of the dual intersection complex. Hence

$$Gr_0^W H_{\lim}^k(Y_0) = H^k(\Gamma).$$

There are four possibilities for the mixed Hodge diamond of the limiting mixed Hodge structures appearing for variations of Hodge structures with $h^{30} = h^{21} = h^{12} = h^{03} = 1$. In the diagrams below the k-th row from the bottom gives the Hodge numbers of Gr_k^W ;

the operator N acts in the vertical direction shifting downwards by two rows. The diagram is symmetric around the central vertical axis (by complex conjugation) and the central horizontal action (by symmetry of the weight filtration). The numbers in each slope = 1 (so SW-NE-direction) row of the diagram have to add up to the corresponding Hodge number, so are all equal to 1 in our case. The cases that arise are:

F-point

In this case N = 0, so this happens if and only if the monodromy is of *finite order*. The limiting mixed Hodge structure is in fact pure of weight three. This happens in the mirror quintic at ∞ , where the monodromy is of order five.

C-point



In this case $N \neq 0$, $N^2 = 0$ and there is a single Jordan block. The pure part Gr_3^W is a rigid Hodge structure with Hodge numbers 1, 0, 0, 1. Furthermore, Gr_4^W and Gr_2^W are one-dimensional and are identified via N. This type appears when a Calabi–Yau threefold aquires one or more ordinary double points, nowadays often called *conifold points*, which explains our name C-type point for it. In the mirror quintic this happens at $t = 1/5^5$. But there are many different kinds of singularties that lead to this mixed Hodge diamond.

K-type point



In this case we also have $N \neq 0$, $N^2 = 0$ but there are two Jordan blocks. In this case the pure part $Gr_3^W = 0$ and Gr_4^W , Gr_2^W are Hodge structures with Hodge numbers 1, 0, 1, which are identified via N. The Hodge structure looks like that of the transcendental part of a K3-surface with maximal Picard number, which explains our name K-point for it. This type of degeneration does not appear in the family of the quintic

mirror, but is common in other examples. The holomorphic three-form is destroyed by the singularties appearing in the fibre.

MUM-point

Here $N^3 \neq 0$ and there is a single Jordan block of maximal size. The Hodge structures Gr_{2k}^W (k = 0, 1, 2, 3) are one-dimensional and necessarily of Tate type. This happens for the quintic mirror at t = 0 and is one of the main defining properties of Calabi–Yau operators.

So at a MUM-point, the resulting mixed Hodge structure is an iterated extension of Tate–Hodge structures. DELIGNE [41] has shown that the instanton numbers n_1, n_2, n_3, \ldots can be seen to encode precisely certain *extension data* attached to the variation of Hodge structures near the MUM-point.

Now it is very well possible that in a family of Calabi–Yau threefolds no MUM-points appear. In [133] first examples were given and in [63] a further example was described, but the corresponding Picard–Fuchs equation was of second order. In [35] an example with differential Galois group Sp(4) was given and, in fact, there are many more. ZUDILIN [157] proposed to call such operators *orphans*, as they do not have a MUM. For Calabi–Yau threefolds appearing in one-parameter families without a MUM-point, it is not clear how to approach the problem of constructing a mirror manifold, nor how to extract enumerative information of the mirror manifold using the Picard–Fuchs equation. For this, one will need to understand the information hidden in the extension data near C-type and K-type points.

2.6. **Integrality properties.** So far most of the properties of the differential operator we discussed were purely algebraic and rather easy to arrange for. For Calabi–Yau operators one supplements these by further arithmetic *integrality conditions*. Initially in [7] it was required that the operator has an integral solution, but it is more natural to allow small denominators and ask for *N*-integral solutions.

We will now explain in some detail the reasons for the integrality of the normalised period near a MUM-point in the case it arises as Picard–Fuchs operator of a family of Calabi–Yau varieties defined over \mathbb{Q} . It can be seen as a generalisation of the celebrated *Theorem of Eisenstein*. In the year 1852 EISENSTEIN [51] reported at the meeting of the Königlich Preußische Akademie der Wissenschaften zu Berlin on a curious general property of the power series development of algebraic functions: if the power series

$$\phi(t) = a_0 + a_1 t + a_2 t^2 + \ldots \in \mathbb{Q}[[t]].$$

solves an equation $0 = R(t, \phi(t))$ where $R \in \mathbb{Z}[x, y]$, then only finitely many primes appear in the denominators of the coefficients a_i :

$$\phi(t) \in \mathbb{Z}[\frac{1}{N}][[t]]$$

He mentions the example $\sqrt{1+t}$, where the replacement of t by 4t turns the series into one with integral coefficients, a fact he considers as well-known. On the other hand, in the series expansion of $\log(1+t)$ and e^t any prime appears in a denominator of a coefficient, and the theorem of EISENSTEIN implies the well-known fact that these series are not algebraic.

As usual, there is a prehistory that goes back to EULER. In a letter to GOLDBACH, EULER [56] reported on the counting of the number A_n of ways to decompose an *n*-gon into triangles by drawing diagonals and found

n	3	4	5	6	7	8	9	10
A_n	1	2	5	14	42	132	429	1430

and gave

$$\frac{1-2a-\sqrt{1-4a}}{2aa}$$

as the generating function for this sequence of numbers that nowadays are called the *Catalan-numbers*. In his reply, GOLDBACH expressed his delight in the fact that this square root function apparently has integral coefficients for its expansion. In the next letter, EULER remarks that more generally the expansion of

$$\sqrt[n]{1-n^2a}$$

in powers of a has only integers as coefficients.

EISENSTEIN did not write down a formal proof of his discovery, but indicated that *once* the truth of the statement was recognised, it was easy to show its truth by the method of undetermined coefficients. What did he have in mind?

It was HEINE [79] who gave a proof of a sharpened version of EISENSTEIN's claim. HEINE also remarked that the series

$$1 + \frac{1}{3}t + \frac{1}{3^2}t^2 + \frac{1}{3^9}t^3 + \frac{1}{3^{16}}t^4 + \dots$$

was not excluded by the theorem of EISENSTEIN, but nevertheless was not algebraic.

Definition: A series $\phi(t) \in \mathbb{Q}[[t]]$ is called *N*-integral if there exist $c, N \in \mathbb{N}$ such that $c\phi(Nt) \in \mathbb{Z}[[t]].$

HEINE proved the following statement:

Theorem of Eisenstein: Algebraic series are N-integral.

Another proof was given by HERMITE [81] and there is a very nice proof of the result using the theory of *diagonals* that we explain now.

Recall that the diagonal of

$$f = \sum a_{k_1 k_2 \dots k_n} x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} \in \mathbb{Q}[[x_1, x_2, \dots, x_n]]$$

is the power series

$$\Delta_n(f) := \sum_{k=0}^{\infty} a_{kk\dots k} t^k \in \mathbb{Q}[[t]].$$

In this way we obtain a \mathbb{Q} -linear diagonalisation map

$$\Delta_n : \mathbb{Q}[[x_1, x_2 \dots, x_n]] \longrightarrow \mathbb{Q}[[t]], \ f \mapsto \Delta_n(f)$$

We can consider the set of rational functions R_n

$$\frac{P(x_1, x_2, \dots, x_n)}{Q(x_1, x_2, \dots, x_n)}, \quad P, Q \in \mathbb{Q}[x_1, x_2, \dots, x_n]$$

(with $Q(0) \neq 0$) that admit a power series expansion and we say that a power series is an *n*-diagonal, if it is the diagonal of such a rational function in *n* variables, that is, if it belongs to

$$\Delta_n(R_n) \subset \mathbb{Q}[[t]] .$$

There is an obvious notion of N-integrality for series in many variables. Rational functions (with rational coefficients) in many variables are obviously N-integral: if we take P and Q with integral coefficients, then we can take the denominator of P(0)/Q(0) as N. As diagonals of N-integral series are clearly N-integral, we see that all n-diagonals are in fact N-integral for some N.

Theorem: (Fürstenberg [59])¹ The 2-diagonals of rational functions are precisely the algebraic series.

This theorem thus provides a natural proof of Eisensteins theorem. Let us indicate the proof. If $F(x, y) \in R_2$, then one can write

$$\phi(t) := \Delta_2(F) = \frac{1}{2\pi i} \int_{\gamma} F(\zeta, \frac{t}{\zeta}) \frac{d\zeta}{\zeta}.$$

The cycle γ encloses some the poles of F on the Riemann-surface given by xy = t, so evaluating the integral by residues shows that $\phi(t)$ indeed is an algebraic function. Conversely, if a series $\phi(t)$ solves $R(t, \phi(t)) = 0$, where $R(x, y) \in \mathbb{Z}[x, y]$, R(0, 0) = 0, $\partial_y R(0, 0) \neq 0$, then it is a nice exercise to show that

$$\phi(t) = \Delta_2(F(x,y)),$$

where

$$F(x,y) = y^2 \frac{\partial_y R(xy,y)}{R(xy,y)},$$

There is a generalisation of this result to more variables:

Theorem: (Denef and Lipschitz [43]) The diagonal of algebraic power series in n variables is the diagonal of a rational function in 2n variables.

From the proof of EISENSTEIN's theorem we see that the diagonalisation map has a natural interpretation in terms of residues and integration over a vanishing cycle as was

$$\phi(t) = \sum_{n=0}^{\infty} t^{p^n},$$

which satisfies the equation

$$\phi(t) = t + \phi(t)^p,$$

hence is an algebraic series. If K is a finite field, then all n-diagonals are algebraic.

¹The main interest of the paper [59] lies, however, in the statement that the situation is completely different over finite fields: many more power series are algebraic, like

pointed out in [40]. Let us consider the following model situation: $X := \mathbb{C}^n$, $S = \mathbb{C}$ and the map

$$p: X \longrightarrow S, (x_1, x_2, \dots, x_n) \mapsto x_1 x_2 \dots x_n = t$$
.

The fibre $X_t := p^{-1}(t), t \neq 0$ is isomorphic to $(\mathbb{C}^*)^{n-1}$ and contains the (n-1)-cycle Γ_t defined by:

$$|x_1| = t_1, |x_2| = t_2, \dots, |x_n| = t_n \subset X_t$$

where t_1, t_2, \ldots, t_n are positive real numbers such that $t_1 t_2 \ldots t_n = |t|$. There is a map

$$\Omega^n_X \longrightarrow \omega_{X/S}; \quad \omega \mapsto Res\Big(\frac{\omega}{x_1 x_2 \dots x_n - t}\Big).$$

Now the statement is

$$\frac{1}{(2\pi i)^n} \int_{\Gamma_t} \operatorname{Res}\left(\frac{hdx_1dx_2\dots dx_n}{x_1x_2\dots x_n - t}\right) = \Delta_n(h)$$

We will now describe a general theorem that implies the *N*-integrality of the invariant period for one-parameter families of Calabi–Yau manifolds near a MUM-point. The theorem has its roots in the work of CHRISTOL, in particular in the following example that can be found in [30].

The power series

$$F(1/2, 1/2, 1; t) = 1 + \left(\frac{1}{2}\right)^2 t + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 t^2 + \dots$$

is the normalised period of the differential form

$$\omega = Res\left(\frac{dxdy}{f}\right) = \frac{dx}{2y}$$

on the standard elliptic curve E_t defined by

$$f(x,y) = y^{2} - x(1-x)(x-t) = 0$$

which for t = 0 aquires a node. The equation can be written in the form

$$t = x - \frac{y^2}{x(1-x)} = \frac{x^2(1-x) - y^2}{x(1-x)} = u \cdot v$$

where

$$u = \frac{x\sqrt{1-x}-y}{\sqrt{x(1-x)}}, \quad v = \frac{x\sqrt{1-x}+y}{\sqrt{x(1-x)}}.$$

Expressed in the coordinates u, v, the form ω transforms to

$$\sqrt{1 - \left(\frac{u+v}{2}\right)^2} \frac{du}{u}$$

so we can represent the normalised period as the diagonal of an algebraic function of two variables:

$$\Delta_2\left(\frac{1}{\sqrt{1-(\frac{u+v}{2})^2}}\right) = F(1/2, 1/2, 1; t)$$

and hence as a diagonal of a rational function of four variables. In fact, in this case, it has even a representation as a diagonal of rational function

$$\frac{4}{4 - (x+y)(1+z)}$$

of three variables.

In [10] (Theorem 2, p.185) this idea is generalised to higher dimensions. Consider a projective family of d-dimensional varieties

$$f: \mathcal{Y} \longrightarrow \mathbb{P}^1, \ Y_t = f^{-1}(t)$$

defined over \mathbb{Q} . We assume that 0 is a MUM-point and that $Gr_0^W H_{lim}^d(Y_0)$ is onedimensional. This implies that for $t \in \mathbb{P}^1 \setminus \Sigma$ the Hodge space $H^{d,0}(Y_t) = H^0(\Omega_{Y_t}^d)$ is one dimensional. Pick a (rational) differential form $\Omega \in H^0(\mathcal{Y}, \omega_{\mathcal{Y}}(*))$ and a cycle

$$\gamma_t \in H_d(Y_t, \mathbb{Z})$$

that is invariant under the monodromy. We can form the period function

$$\phi(t) := \int_{\gamma_t} \omega_t, \quad \omega_t := \operatorname{Res}\left(\frac{\Omega}{f-t}\right)$$

that is defined in a sufficiently small disc around 0. We can write

$$\phi(t) = Cy_0(t)$$

where the normalised period expands as

$$y_0(t) := 1 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots \in \mathbb{Q}[[t]].$$

Theorem: (Christol-André)

The series y(t) is the diagonal of an algebraic function of d + 1-variables.

Proof: We may take a semi-stable model

$$\mathcal{Z} \xrightarrow{\pi} \mathcal{Y} \xrightarrow{f} \Delta$$

of our family over a disc Δ . We let $g := f \circ \pi : \mathbb{Z} \longrightarrow \Delta$ and set $Z_t = g^{-1}(t)$. For $t \neq 0$ we have $Z_t \approx^{\pi} Y_t$. The singular fibre Y_0 is replaced by a reduced normal crossing divisor $D = \bigcup D_i \subset \mathbb{Z}$. We pull back ω to η on \mathbb{Z} and can write the period function as

$$\phi(t) = \int_{\delta_t} \operatorname{Res}_{Z_t}\left(\frac{\eta}{f-t}\right),$$

where the cycle δ_t maps to γ_t via π . From STEENBRINKS construction of the limiting mixed Hodge structure on $H^d_{\text{lim}}(Y_0)$ we obtain from the weight spectral sequence a description $Gr_0^W H^d_{\text{lim}}(Y_0)$ in terms of the intersections of the divisors D_i

$$Gr_0^W H^d_{\lim}(Y_0) = H^d(\Gamma),$$

where Γ is the dual intersection complex of the divisors D_i . By assumption, this space is one-dimensional so in particular, there must be points $m \in \mathcal{Y}$ where d + 1 divisors intersect.

For each such point, we may compare the behaviour of f with the standard model. Let $R = \mathcal{O}_{\mathcal{Y},m}$ the local ring of \mathcal{Y} at one of these points m. The equations defining the d + 1 divisors meeting at m may not belong to R, but in the Henselisation \widehat{R} of R we find elements x_0, x_1, \ldots, x_d such that D_i is locally defined by $x_i = 0$ and we can arrange that the map f is given by

$$(x_0, x_1, \dots, x_n) \mapsto x_0 x_1 \dots x_d = t$$

By considering the points with $|x_i| = t_i$ fixed, $t_0 t_1 \dots t_d = |t|$, we obtain a real *d*-dimensional torus $T_m(t) \subset \mathcal{Y}_t$ that vanishes at the point *m* if $t \to 0$. As the group

 $W_0 H^d_{\text{lim}}(Y_t)$ is supposed to be one-dimensional, all these tori $T_m(t)$ are homologous to a rational multiple of δ_t . Writing the form η in terms of the coordinates x_i we have

$$\eta = h(x_0, \dots, x_d) dx_0 \wedge dx_1 \wedge \dots \wedge dx_d$$

with $h \in \widehat{R}$. So

$$\phi(t) = \int_{\delta_t} \operatorname{Res}\left(\frac{\eta}{g-t}\right) = c \int_{T_m} \operatorname{Res}\left(\frac{hdx_0dx_1\dots dx_d}{x_0x_1\dots x_d-t}\right) = c\Delta_{d+1}(h)$$

So the normalised period is indeed the diagonal of an algebraic function in d + 1 variables.

Combining this with the theorem of DENEF and LIPSCHITZ we get:

Corollary: The normalised monodromy invariant period near a MUM-point of a Calabi–Yau d-fold is an 2(d + 1)-diagonal, hence is N-integral.

In particular, for Calabi–Yau threefolds, the period y_0 is an 8-diagonal!

In the context of Calabi–Yau operators, one asks also for the integrality of the mirror map q(t). In some cases integrality of the mirror map has been shown by LIAN and YAU [112], KRATTENTHALER [106] and DELAYGUE [42] using purely number theoretic methods. But for the majority of cases, the integrality of the mirror map remains unproven.

The integrality of the n_d is much deeper. In an A-incarnation, although supposed to count rational degree d curves, n_d is defined in terms of Gromov–Witten invariants, so are a priori only in \mathbb{Q} . So the integrality of the n_d , which was the biggest selling point of [27], is in the end the most mysterious aspect of the calculation. The conjectural duality between Gromov–Witten theory and Donaldson–Thomas theory [119] would provide a natural explanation. The recent paper [93] provides a proof of the integrality of the n_d in case the operator has an A-incarnation. It uses purely symplectic methods.

In the case of a B-incarnation one can use the link between the action of Frobenius and the Yukawa coupling discovered by KONTSEVICH, SCHWARZ and VOLOGDSKY [102], [147]. A claim was made in [152] that in the geometrical case the instanton numbers are N-integral, where the N relates to primes of bad reduction in the semi-stable reduction.

In the thesis of BOGNER [21] one finds the following interesting operator:

$$\mathcal{P} := \Theta^4 - 8t(2\Theta + 1)^2(5\Theta^2 + 5\Theta + 2) + 192t^2(2\Theta + 1)(3\Theta + 2)(3\Theta + 4)(2\Theta + 3).$$

The operator has an integral solution

 $y_0(t) = 1 + 16t + 576t^2 + 25600t^3 + 1220800t^4 + \dots,$

integral mirror map

 $q = t + 40t^2 + 1984t^3 + 106496t^4 + \dots,$

integral Yukawa coupling

$$K(q) = 1 + 8q - 5632q^3 - 456064q^4 - 17708032q^5 + \dots$$

but the corresponding n_d 's are not integral: n_p has denominator p^2 for

$$p = 3, 5, 7, 11, 13, 17, 19, \dots$$

This is rather puzzling. The integrality of solution and mirror map clearly indicate that we have a rank four Calabi–Yau motive and one would expect the general arguments for the integrality of [152] to be applicable, but apparently they are not. Maybe there is a different scaling of the coordinate that repairs this defect, but up to now we have been unable to find it.

Questions:

(i) Is there a proof of the integrality of the mirror map along the same lines as the proof of integrality of the normalised period y_0 ? The higher dimensional strata in the divisor D clearly will be relevant.

(ii) Constant terms series of Laurent polynomials are special cases of diagonals. The cases specially relevant to Calabi–Yau periods are the reflexive ones, and more generally those with a single interior point. Is there a similar theory of reflexive diagonals?

(iii) For the constant term of the powers of a Laurent polynomial whose Newton polyhedron contains a single interior point there are so-called *Dwork congruences*, see [137] and [121]. Is there an analogue for diagonals?

2.7. Monodromy conjecture. The monodromy group $\Gamma \subset Sp_4(\mathbb{Z})$ appearing in oneparameter families of Calabi–Yau threefolds is largely mysterious. The paper [27] suggested that this group has infinite index in $Sp_4(\mathbb{Z})$, but the arguments given were insufficient. Only recently [26] this was proven to be the case for seven of the 14 hypergeometric cases. Somewhat surprisingly, in the other seven cases the monodromy group turned out to have finite index, [144, 145]. The situation for other one-parameter families is under active study, [83]. BOGNER and REITER determined all symplectically rigid local systems of rank four [23]; strong B-realisations have been described recently in detail in [48].

In the paper [27] the explicit analytic continuation of solutions for the operator were derived from the classical *Barnes integral representation*, which is tied to the hypergeometric nature of the operator. One of the striking features is the appearance of the number $\zeta(3)$ in combination with the Euler number of the quintic. For this, the four loop correction to the sigma-model was made responsible. In that calculation, $\zeta(3)$ enters via the third derivative of the Γ -function at 1.

The paper [86] contained a general study of complete intersections in products of projective spaces. The relevant holomorphic period function has nice expansions, whose coefficients are expressible as a quotient of products of factorials, like the $a(n) = (5n)!/(n!)^5$ appearing in the case of the mirror quintic. The solutions with log-terms can be obtained from the Frobenius method, i.e. as the derivatives of $\sum a(n+\rho)x^{n+\rho}$ at $\rho = 0$. If we replace all factorials by Γ using the relation $x! = \Gamma(1+x)$, we are naturally led to consider the powerseries expansion

$$\frac{\Gamma(1+5x)}{\Gamma(1+x)^5} = 1 + \frac{5}{3}\pi^2 x^2 - 40\zeta(3)x^3 + \dots$$

If we set

$$h := \frac{x}{2\pi i}$$

this can be written in the form

$$\frac{\Gamma(1+5x)}{\Gamma(1+x)^5} = \frac{1}{5}(5-\frac{50}{24}h^2-200\frac{\zeta(3)}{(2\pi i)^3}h^3+\ldots).$$

Lo and behold, the coefficients of the expansion in h contain the characteristic numbers of the quintic X:

$$\int_X H \cdot H \cdot H = 5, \quad \int_X c_1(X) \cdot H = 0, \quad \int_X c_2(X) \cdot H = 50, \quad \int_X c_3(X) = -200 \quad !$$

In the paper [86] these *remarkable identities* were observed to hold for all complete intersections in products of projective spaces and a version of it was generalised to the toric setting in [87]. Inspired by these facts and the formal similarity between the Chern polynomial of the quintic

series
$$\frac{\frac{(1+h)^5}{(1+5h)}}{\frac{\Gamma(1+5x)}{\Gamma(1+x)^5}}$$

and the above series

LIBGOBER tried to find a general formulation of this relationship and introduced the Hirzebruch genus associated to the power series $\frac{1}{\Gamma(1+x)}$.

In [54] we started computing monodromies numerically and discovered the systematic apearance of $\zeta(3)$ for general (fourth order) Calabi–Yau operators. This led to the following general conjecture:

Conjecture 1

The monodromy matrices for a Sp(4)-Calabi-Yau operator with respect to the scaled Frobenius basis u_0, u_1, u_2, u_3 have entries in

$$\mathbb{Q}[\lambda]$$

where

$$\lambda := \frac{\zeta(3)}{(2\pi i)^3}.$$

Recall that the monodromy around 0 on the solution space

$$H_0 = \langle u_0, u_1, u_2, u_3 \rangle$$

is always represented by the matrix

$$T_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1/2 & 1 & 1 & 0 \\ 1/6 & 1/2 & 1 & 1 \end{pmatrix}.$$

Furthermore, one can show that the skew-symmetric form

$$\langle -, - \rangle : H_0 \times H_0 \longrightarrow \mathbb{C}$$

determined by the conditions

$$\langle u_0, u_3 \rangle = -\langle u_1, u_2 \rangle = \langle u_2, u_1 \rangle = -\langle u_3, u_0 \rangle$$

is monodromy invariant.

A Calabi–Yau operator is called a *conifold operator* if the exponents around the singular point c nearest to 0 are 0, 1, 1, 2 and the monodromy around c is a *symplectic reflection*. So the monodromy T_c around c can be described in terms of a vector $S \in H_0$ via the formula

$$T_c: H_0 \longrightarrow H_0; \quad v \mapsto v - \langle v, S \rangle S.$$

We call S the reflection vector.

For example, the operator of Candelas is a conifold operator with $c = 1/5^5$ and the vector S in the u-base is

$$(5, 0, \frac{25}{12}, -200\lambda)^T$$

where we scale the skew-form by putting

$$\langle u_0, u_3 \rangle = \frac{1}{5}.$$

Now note that for the quintic Calabi–Yau $X \subset \mathbb{P}^4$ and $H \in H^2(X)$ the hyperplane class we have as mentioned before:

$$5 = \int_X H \cdot H \cdot H,$$

$$0 = \int_X c_1(X) \cdot H \cdot H,$$

$$50 = \int_X c_2(X) \cdot H,$$

$$\cdot 200 = \int_X c_3(X).$$

One can verify for all cases where the operator arises from an A-incarnation of a Calabi– Yau threefold X with $h^{11} = 1$, the operator indeed is a conifold operator, and that the corresponding reflection vector is of the form

$$(d, 0, \frac{c}{24}, e\lambda)^T, \quad \langle u_0, u_3 \rangle = \frac{1}{d}$$

and thus determines the characteristic numbers of X.

$$d = \int_X H \cdot H \cdot H,$$

$$0 = \int_X c_1(X) \cdot H \cdot H,$$

$$c = \int_X c_2(X) \cdot H,$$

$$e = \int_X c_3(X).$$

And even more, the reflection vectors of all further conifold operators of the AESZ-list appear to be of this form, where d, c, e are *integers*. From this one may conjecture the existence of Calabi–Yau threefolds with the given characteristic numbers. This was the main idea of the paper [54]. In the meantime, a few of these conjectured Calabi–Yau varieties X have indeed been found, but there is still a big gap.

There is a beautiful interpretation of the monodromy conjecture in terms of homological mirror symmetry supplemented by the Γ -class introduced by LIBGOBER [114] discussed above and developed further by KONTSEVICH, KATZARKOV, PANTEV [101] and IRITANI [94]. We will sketch now that intriguing line of reasoning that, needless to say, is largely conjectural in general.

According to KONTSEVICH [100], mirror symmetry should be understood as an equivalence of categories

$$D(X) \xrightarrow{Mir} F(Y)$$

and for the Calabi–Yau hypersurfaces $Y \subset \mathbb{P}^n$ a version of this has recently been proven by SHERIDAN [142]. The category on the left is $D^b(Coh(X))$, the bounded derived category of coherent sheaves on X, on the right we have $D^{\pi}(Fuk(Y))$, the derived Fukaya category of Y. The objects in this category are represented by Lagrangian cycles (with local systems on them) in Y and the category only depends on the symplectic manifold underlying Y_t . The Hom-spaces in this category are given by Floer homology HF(L, L') groups; its Euler characteristic is just the intersection product of the corresponding cycles:

$$\langle L, L' \rangle := [L] \cdot [L']$$

On the left hand side, the Euler characteristic of the Hom-spaces between \mathcal{E} and \mathcal{F} in D(X) is the Euler pairing

$$\langle \mathcal{E}, \mathcal{F} \rangle := \sum_{i} (-1)^{i} \dim Ext^{i}(\mathcal{E}, \mathcal{F}).$$

Under mirror symmetry these should correspond:

$$\langle \mathcal{E}, \mathcal{F} \rangle = \langle Mir(\mathcal{E}), Mir(\mathcal{F}) \rangle.$$

In the SYZ-picture of mirror symmetry, the spaces X and Y are related by T-duality: both X and Y are supposed to have the structure of dual three-torus fibrations over a common base. In [148] it is argued that under Mir the structure sheaf \mathcal{O}_p of a point is mapped to a lagrangian torus **T** (with local system on it) in Y, and the structure sheaf \mathcal{O}_X is mapped to a lagrangian sphere **S**:

$$Mir(\mathcal{O}_X) = \mathbf{S}, \quad Mir(\mathcal{O}_p) = \mathbf{T}$$

and indeed

$$\langle \mathcal{O}_X, \mathcal{O}_p \rangle = 1 = \langle \mathbf{S}, \mathbf{T} \rangle.$$

But there is a certain asymmetry: if we map objects of D(X) to $H^{ev} := \bigoplus_k H^{2k}(X)$ via the Chern character, we can express the Euler pairing as

$$\int_X ch(\mathcal{E}^*) ch(\mathcal{F}) T d(X).$$

Objects of F(Y) represented by lagrangian cycles map directly to $H^{odd} = H^3(Y)$ and the pairing $\langle L, L' \rangle$ is just given as an intersection number; no tangential information like Td(X) comes in. To overcome this asymmetry, one modifies the Chern character by slipping in a sort of square root of the Todd class. Recall that the Todd class is the characteristic class coming from the power series

$$\frac{x}{1-e^{-x}}.$$

The identity $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ for the Γ -function is equivalent to

$$\Gamma(1 + \frac{x}{2\pi i})\Gamma(1 - \frac{x}{2\pi i}) = e^{x/2} \frac{x}{1 - e^{-x}}.$$

Now introduce the Γ -class as the characteristic class belonging to power series expansion of

$$\Gamma(1+\frac{x}{2\pi i}) = \exp(-\frac{\gamma}{2\pi i}x + \sum_{k=2}^{\infty}\frac{\zeta(k)}{(2\pi i)^k}\frac{x^k}{k})$$

So one puts:

$$\Gamma(T_X) := \prod_i \Gamma(1 + \frac{\xi_i}{2\pi i})$$

where the ξ_i are the chern roots of T_X . Then one can write

$$\langle \mathcal{E}, \mathcal{F} \rangle = (\psi(\mathcal{E})^* \cdot \psi(\mathcal{F})),$$

where

$$\psi(\mathcal{E}) := \Gamma(T_X) \cup ch(\mathcal{F})$$

and the operation * multiplies a component in H^{2k} by $(-1)^k$. So we are supposed to get a commutative diagram

where mir is the cohomological mirror map. (The same diagram is discussed in this context at various places in the literature, see e.g. [85]).

The geometric monodromy of the family Y_t , $t \in \mathbb{P}^1 \setminus \Sigma$, can be realised symplectically and acts as auto-equivalences on F(Y). Via mirror symmetry there should be a corresponding action on D(X). And indeed, KONTEVICH conjectured that the monodromy around the MUM-point corresponds to the auto-equivalence $\mathcal{O}(H) \otimes :$ $D(X) \longrightarrow D(X)$, whereas the *Seidel-Thomas twist* in the spherical object \mathcal{O}_X corresponds to the symplectic Dehn twist along the sphere **S**.

But that implies that the reflection vector for the monodromy around the conifold point should be equal to

$$S = mir(\psi(\mathcal{O}_X)) = \phi(\mathbf{S}) \in H^{odd}(Y).$$

We will identify the space H^{odd} with the solution space H_0 , spanned by the scaled Frobenius basis. Now we can work out everything!

In the basis 1, H, H^2, H^3 for $H^{ev}(X)$ the operation of $\mathcal{O}(H)\otimes$ is mapped via the Chern character to multiplication with e^h , so is represented by exactly the same matrix as T_0 in the *u*-basis. Therefore, it is natural to put

$$mir(h^k) = \alpha u_k$$

where α is to be determined. The element $\psi(\mathcal{O}_X) = \Gamma(T_X)$ is computed to be

$$\psi(\mathcal{O}_X) = 1 - \lambda_2 c_2(X) - \lambda_3 c_3(X),$$

where

$$\lambda_k := \frac{\zeta(k)}{(2\pi i)^k}.$$

Writing this in terms of the basis $1,h,h^2,h^3$ and applying mir we find for the reflection vector

$$S := mir(\psi(\mathcal{O}_X)) = \frac{\alpha}{d}(du_0 + \frac{c}{24}u_2 - \lambda_3 eu_3),$$

where

$$d = \int_X H \cdot H \cdot H, \quad c = \int_X c_2(X) \cdot H, \quad e = \int_X c_3(X).$$

So we take $\alpha = d$. Furthermore, $mir(\psi(\mathcal{O}_p))$ represents the cohomology class of the torus in $H^{odd} = H_0$. We find $\phi(\mathcal{O}_p) = \frac{1}{d}H^3$, so that

$$T := mir(\phi(\mathcal{O}_p)) = \frac{\alpha}{d}u_3 = u_3(=y_0)$$

As we are supposed to have $\langle S, T \rangle = 1$, we see that the right scaling of the skew form indeed is obtained by putting

$$\langle u_0, u_3 \rangle = -\langle u_1, u_2 \rangle = \frac{1}{d} = \langle u_2, u_1 \rangle = -\langle u_3, u_0 \rangle.$$

So, miraculously, everything fits ² and completely explains the structure of the conifold reflection vector in the Frobenius basis. Usually a Calabi–Yau operator has more conifold points and the corresponding reflection vectors should arise from other spherical objects in D(X). As before, we obtain vectors of the same shape

$$\frac{1}{d}(d,a,\frac{c}{24},e\lambda_3)^T$$

but now usually $a \neq 0$. For more complicated monodromy transformations we do not have a real argument, but there is little reason to doubt the general principle.

If the monodromy is not in Sp(4), there usually appear algebraic numbers and the monodromy matrices appear to be contained in

$$\mathbb{Q}[\lambda].$$

As an example, take the operator number 245 from the AESZ-list [3] mentioned earlier:

$$\Theta^{4} - t(216\theta^{4} + 396\Theta^{3} + 366\Theta^{2} + 168\Theta + 30) + 36(3\Theta + 2)^{2}(6\Theta + 7)^{2}$$

with Riemann symbol

$$\left\{ \begin{array}{cccc} 0 & 1/108 & \infty \\ \hline 0 & 0 & 2/3 \\ 0 & 1/6 & 2/3 \\ 0 & 1 & 7/6 \\ 0 & 7/6 & 7/6 \end{array} \right\}.$$

For the matrix of the monodromy around 1/108 in the Frobenius basis we find

$$\frac{\sqrt{3}+i}{2} \begin{pmatrix} 1/2\sqrt{3}-72\sqrt{3}\lambda & -\sqrt{3} & 0 & 18\sqrt{3} \\ -1/6\sqrt{3} & 1/2\sqrt{3} & -3\sqrt{3} & 0 \\ 4\sqrt{3}\lambda & 1/12\sqrt{3} & 1/2\sqrt{3} & -\sqrt{3} \\ -\frac{1}{72}\left(1+20736\lambda^2\right)\sqrt{3} & -4\sqrt{3}\lambda & -1/6\sqrt{3} & 1/2\sqrt{3}+72\sqrt{3}\lambda \end{pmatrix},$$

which indeed has order six.

One can generalise this to higher order operators:

Conjecture 2

The monodromy matrices for a Calabi–Yau operator of order n + 1 with respect to the scaled Frobenius basis have entries in

$$\overline{\mathbb{Q}}[\lambda_2, \lambda_3, \ldots, \lambda_n],$$

where

$$\lambda_k := \frac{\zeta(k)}{(2\pi i)^k}.$$

This conjecture has been verified numerically for almost all Calabi–Yau operators of the list. For those cases that can be related to specialisations of hypergeometric functions of more variables one can in principle *prove* these numerical results. For details

²This I realised after a talk by Kontsevich (Vienna, 2008), where he explained the Γ -class.

we refer to the thesis of HOFMANN, [82].

It turns out that it is natural to formulate conjectures about the Γ -class in the context of Fano manifolds, as was done in the beautiful paper [61]. The so-called Γ -conjectures formulated there can be motivated from mirror symmetry [62] and they imply the above monodromy conjecture in those cases where the Calabi–Yau manifold has an Aincarnation as a complete intersection in a Fano-manifold for which the Γ -conjectures are proven. In the paper [68] the Γ -conjectures were verified for the case of Fano threefolds of Picard rank equal to one.

In this overview paper we have touched upon various aspects of Calabi–Yau operators. We had to leave out several important topics, most notably be *p*-adic story, which involves the Dwork congruences [137, 121], the computations of the local L-factors [136, 135] and the *p*-adic analogue of the Γ -conjectures that lead to the appearance of the *p*-adic analogue of $\zeta(3)$. Also higher genus instanton numbers for Calabi–Yau operators were completely left out of this account.

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